QUASI-HOMOGENEOUS LINEAR SYSTEMS ON $\mathbb{P}^2$ WITH BASE POINTS OF MULTIPLICITY 6

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Abstract. In this paper we prove the Harbourne-Hirschowitz conjecture for quasi-homogeneous linear systems of multiplicity 6 on $\mathbb{P}^2$. For the proof we use the degeneration of the plane by Ciliberto and Miranda and results by Laface, Seibert, Ugaglia and Yang. As an application we derive a classification of the special systems of multiplicity 6.

1. Introduction

A classical problem in algebraic geometry is the dimensionality problem for plane curves, which can be formulated as follows. Given finitely many general points of the projective plane with assigned multiplicities and a number d, determine the dimension of the linear system of curves of degree d having at the given points at least the assigned multiplicities. More precisely, the problem is to classify all systems which fail to have the expected dimension (see [C00] for some remarks on the history of this problem and its geometric meaning). Harbourne and Hirschowitz conjecture that these special systems are precisely the (-1)-special systems. In this paper, we give a complete list of the (-1)-special systems in the case in which the assigned multiplicity is 6 at all but one of the given points. Our main result is the proof of the Harbourne-Hirschowitz conjecture in this case.

We proceed along the following lines. In Section 2 we introduce the necessary notation and give a precise statement of the Harbourne-Hirschowitz conjecture. In Section 3 we present a list of the (-1)-special linear systems in our case. Its completeness is proved in Section 4. In Section 5 we review the degeneration of the plane by Ciliberto and Miranda. This method is the key tool in our proof of the main result which is given in the final two sections.

2. The Harbourne-Hirschowitz conjecture

We work over the complex numbers and choose $n+1$ general points $p_0, p_1, \ldots, p_n$ in $\mathbb{P}^2$, the projective plane over that field.

2.1 Notation
We write $\mathcal{L} = \mathcal{L}(d, m_0, m_1, \ldots, m_n) \subset \mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)))$ for the linear system of all curves of degree d in $\mathbb{P}^2$ having multiplicity at least m_i at p_i for all i. We denote by $\ell(\mathcal{L})$ its projective dimension.

Let $\mathbb{P}'$ be the blow-up of $\mathbb{P}^2$ at $p_0, p_1, \ldots, p_n$. By H we denote the pull-back of a line in $\mathbb{P}^2$ and by E_i the exceptional divisor over p_i. The dimension of $\mathcal{L}$ is the same as the dimension of $|D|$ on $\mathbb{P}'$ with $D = dH - m_0E_0 - m_1E_1 - \ldots - m_nE_n$. Using cohomology on $\mathbb{P}'$, we have

$$\ell(\mathcal{L}) = h^0(\mathcal{O}_{\mathbb{P}'}(D)) - 1.$$

Therefore we have by Riemann-Roch

$$\ell(\mathcal{L}) = \frac{D.(D - K_{\mathbb{P}'})}{2} + h^1(\mathcal{O}_{\mathbb{P}'}(D)) - h^2(\mathcal{O}_{\mathbb{P}'}(D)) + \chi(\mathcal{O}_{\mathbb{P}'}) - 1$$

($K_{\mathbb{P}'}$ denotes the canonical divisor on $\mathbb{P}'$). Since the arithmetic genus of $\mathbb{P}'$ is zero, Serre duality implies

$$\ell(\mathcal{L}) = \frac{D.(D - K_{\mathbb{P}'})}{2} + h^1(\mathcal{O}_{\mathbb{P}'}(D)).$$

2.2 Definition
We define the virtual dimension $v(\mathcal{L})$ of $\mathcal{L}$ as follows:

$$v(\mathcal{L}) = \frac{D.(D - K_{\mathbb{P}'})}{2}.$$

We define the expected dimension to be

$$e(\mathcal{L}) = \max\{-1, v(\mathcal{L})\}.$$

As $v(\mathcal{L}) = \frac{d(d+3)}{2} - \sum_{i=0}^{n} \frac{m_i(m_i+1)}{2}$, one sees that the expected dimension is the one we obtain if all conditions imposed on the base points are independent.

We define $\mathcal{L}$ to be special or non-regular if

$$\ell(\mathcal{L}) > e(\mathcal{L}),$$

otherwise we call $\mathcal{L}$ non-special or regular.

We recall some definitions from [CM98]:

2.3 Definition ((-1)-special systems)
Let $\mathcal{A}$ in $\mathbb{P}^2$ be an irreducible curve such that its strict transform $\tilde{\mathcal{A}}$ in $\mathbb{P}'$ is rational and smooth. Then $\mathcal{A}$ is a (-1)-curve if the self-intersection number

$$\tilde{\mathcal{A}}^2 = -1.$$

By $\mathcal{L}.\mathcal{A}$ we denote the intersection number $D.\tilde{\mathcal{A}}$ on $\mathbb{P}'$.

The linear system $\mathcal{L}$ is called (-1)-special if

- *there exist $\mathcal{A}_1, \dots, \mathcal{A}_t$ (-1)-curves with $\mathcal{L}.\mathcal{A}_i = -n_i$ such that $n_i \geq 1$ for all i,*
- *there is an j with $n_j \geq 2$ and*
- *the residual system $\mathcal{M} = \mathcal{L} - \sum_{i=0}^{t} n_i \mathcal{A}_i$ has $v(\mathcal{M}) \geq 0$.*

The main conjecture can be formulated as follows:

2.4 Conjecture (Harbourne-Hirschowitz)
A linear system $\mathcal{L} = \mathcal{L}(d, m_0, m_1, \dots, m_n)$ is special if and only if it is (-1)-special.

It is easy to see that a (-1)-special system $\mathcal{L}$ is special because

$$v(\mathcal{L}) = \frac{\mathcal{L}.(\mathcal{L} - K_{\mathbb{P}'})}{2} = \frac{(\mathcal{M} + n\mathcal{A}).(\mathcal{M} + n\mathcal{A} - K_{\mathbb{P}'})}{2}.$$

Since $\mathcal{A}.K_{\mathbb{P}'} = -1$ by the rationality of $\tilde{\mathcal{A}}$, this implies

$$v(\mathcal{L}) = v(\mathcal{M}) + \frac{-n^2 + n}{2} \leq \ell(\mathcal{L}) + \frac{-n^2 + n}{2}.$$

Therefore the opposite direction of the Harbourne-Hirschowitz conjecture is the non-trivial one. It states that every special system $\mathcal{L}$ has fixed multiple (-1)-curves. Proving the conjecture leads to an answer of the dimensionality problem.

2.5 Remark
We give a list of results on the conjecture. In fact we use all of them in several ways for the proof of our main theorem.

We write $\mathcal{L} = \mathcal{L}(d, m_0^{b_0}, m_1^{b_1}, \dots, m_r^{b_r})$ if $\mathcal{L}$ has precisely b_i base points of multiplicity m_i for $i = 0, \dots, r$. With this notation the conjecture holds if

- $b_0 + \ldots + b_r \leq 9$ [H89],
- $\mathcal{L} = \mathcal{L}(d, m^n)$ *(call it homogeneous of multiplicity m) and* $m \leq 12$ [CM00],
- $\mathcal{L} = \mathcal{L}(d, m_0, m^n)$ *(call it quasi-homogeneous of multiplicity m) and* $m \leq 3$ [CM98],
- $\mathcal{L} = \mathcal{L}(d, m_0, 4^n)$ [S99] *and* [L99],
- $\mathcal{L} = \mathcal{L}(d, m_0, 5^n)$ [LU02] *or*
- *all multiplicities are bounded by* 6*, i.e.* $m_i \leq 6$ *for* $i = 0, 1, \ldots, n$ [Y03].

3. Main Results

Our main result is a proof of the Harbourne-Hirschowitz conjecture in the quasi-homogeneous case of multiplicity 6:

Theorem A (Main Theorem)
A system $\mathcal{L}(d, m_0, 6^n)$ is special if and only if it is (-1)-special.

We give the proof within an extra section. For the proof we need the following classification:

Theorem B (Classification of (-1)-special systems $\mathcal{L}(d, m_0, 6^n)$)
The following is a complete list of all (-1)-special systems $\mathcal{L}(d, m_0, 6^n)$.

$d - m_0$	*system*	$v(\mathcal{L})$	$\ell(\mathcal{L})$	
0	$\mathcal{L}(d, d, 6^n)$	$-21n + d$	$-6n + d$	$d \geq 6n \geq 6$
1	$\mathcal{L}(d, d-1, 6^n)$	$-21n + 2d$	$-11n + 2d$	$d \geq \frac{11}{2}n \geq \frac{11}{2}$
2	$\mathcal{L}(10e, 10e-2, 6^{2e})$	$-12e - 1$	0	$e \geq 1$
	$\mathcal{L}(d, d-2, 6^n)$	$-21n + 3d - 1$	$-15n + 3d - 1$	$d \geq \frac{1+15n}{3} \geq \frac{16}{3}$
3	$\mathcal{L}(9e, 9e-3, 6^{2e})$	$-6e - 3$	0	$e \geq 1$
	$\mathcal{L}(9e+1, 9e-2, 6^{2e})$	$-6e + 1$	2	$e \geq 1$
	$\mathcal{L}(d, d-3, 6^n)$	$-21n + 4d - 3$	$\geq -18n + 4d - 3$ $=$ *if* $d \neq \frac{9n}{2} + 1$ *or* n *odd*	$d \geq \frac{18n+3}{4} \geq \frac{21}{4}$
4	$\mathcal{L}(8e, 8e-4, 6^{2e})$	$-2e - 6$	0	$e \geq 1$
	$\mathcal{L}(8e+1, 8e-3, 6^{2e})$	$-2e - 1$	2	$e \geq 1$
	$\mathcal{L}(8e+2, 8e-2, 6^{2e})$	$-2e + 4$	5	$e \geq 1$
	$\mathcal{L}(d, d-4, 6^n)$	$-21n + 5d - 6$	$\geq -20n + 5d - 6$ $=$ *if* $d \neq 4n + 2$ *or* n *odd*	$d \geq \frac{20n+6}{5} \geq \frac{26}{5}$
5	$\mathcal{L}(7e, 7e-5, 6^{2e})$	-10	0	$e \geq 1$
	$\mathcal{L}(7e+1, 7e-4, 6^{2e})$	-4	2	$e \geq 1$
	$\mathcal{L}(7e+2, 7e-3, 6^{2e})$	2	5	$e \geq 1$
	$\mathcal{L}(7e+3, 7e-2, 6^{2e})$	8	9	$e \geq 1$
6	$\mathcal{L}(6e, 6e-6, 6^{2e})$	-15	0	$e \geq 1$

$d-m_0$	*system*	$v(\mathcal{L})$	$\ell(\mathcal{L})$	
	$\mathcal{L}(6e+1,6e-5,6^{2e})$	-8	2	$e\geq 1$
	$\mathcal{L}(6e+2,6e-4,6^{2e})$	-1	5	$e\geq 1$
	$\mathcal{L}(6e+3,6e-3,6^{2e})$	6	9	$e\geq 1$
	$\mathcal{L}(6e+4,6e-2,6^{2e})$	13	14	$e\geq 1$
7	$\mathcal{L}(5e+2,5e-5,6^{2e})$	$-2e-5$	$-2e+5$	$2\geq e\geq 1$
	$\mathcal{L}(5e+3,5e-4,6^{2e})$	$-2e+3$	$-2e+9$	$4\geq e\geq 1$
	$\mathcal{L}(5e+4,5e-3,6^{2e})$	$-2e+11$	$-2e+14$	$7\geq e\geq 1$
	$\mathcal{L}(5e+5,5e-2,6^{2e})$	$-2e+19$	$-2e+20$	$10\geq e\geq 1$
8	$\mathcal{L}(4e+4,4e-4,6^{2e})$	$-6e+8$	$-6e+14$	$2\geq e\geq 1$
	$\mathcal{L}(4e+5,4e-3,6^{2e})$	$-6e+17$	$-6e+20$	$2\geq e\geq 1$
	$\mathcal{L}(4e+6,4e-2,6^{2e})$	$-6e+26$	$-6e+27$	$4\geq e\geq 1$
	$\mathcal{L}(10,2,6^3)$	-1	2	
	$\mathcal{L}(24,16,6^9)$	-1	0	
9	$\mathcal{L}(3e+6,3e-3,6^{2e})$	$-12e+24$	$-12e+27$	$2\geq e\geq 1$
	$\mathcal{L}(3e+7,3e-2,6^{2e})$	$-12e+34$	$-12e+35$	$2\geq e\geq 1$
	$\mathcal{L}(9,0,6^3)$	-9	0	
	$\mathcal{L}(10,1,6^3)$	1	4	
	$\mathcal{L}(14,5,6^5)$	-1	0	
	$\mathcal{L}(18,9,6^7)$	-3	0	
10	$\mathcal{L}(2e+8,2e-2,6^{2e})$	$-20e+43$	$-20e+44$	$2\geq e\geq 1$
	$\mathcal{L}(10,0,6^3)$	2	5	
	$\mathcal{L}(14,4,6^5)$	4	5	
11	$\mathcal{L}(13,2,6^5)$	-4	2	
	$\mathcal{L}(14,3,6^5)$	8	9	
12	$\mathcal{L}(12,0,6^5)$	-15	0	
	$\mathcal{L}(13,1,6^5)$	-2	4	
	$\mathcal{L}(14,2,6^5)$	11	12	
13	$\mathcal{L}(13,0,6^5)$	-1	5	
	$\mathcal{L}(14,1,6^5)$	13	14	
14	$\mathcal{L}(14,0,6^5)$	14	15	

4. The Classification

In the paper [CM98] of Ciliberto and Miranda a lot of classification work has been done which we can apply to our problem. Ciliberto and Miranda introduced two notions which we recall now to use their results.

Let $\mathcal{L}$ be a linear system of plane curves with general multiple base points as above. Then $\mathcal{L}$ is a *quasi-homogeneous (-1)-class* if $\mathcal{L}=\mathcal{L}(d,m_0,m^n)$, on $\mathbb{P}'$ the self-intersection number $\mathcal{L}.\mathcal{L}=-1$ and the arithmetic genus

$$g_{\mathcal{L}}=\frac{\mathcal{L}^2+\mathcal{L}.K_{\mathbb{P}'}}{2}+1=0.$$

As $v(\mathcal{L})=\mathcal{L}^2-g_{\mathcal{L}}+1$, these systems are never empty.

In this case, if $\mathcal{A}$ is a (-1)-curve such that $\mathcal{A}\in\mathcal{L}$ then by $\mathcal{L}.\mathcal{A}=-1$ and the irreducibility of $\mathcal{A}$, we have $\mathcal{L}=\{\mathcal{A}\}$. So we can identify (-1)-curves and quasi-homogeneous (-1)-classes and write $\mathcal{A}=\mathcal{L}$. Ciliberto and Miranda proved that such a (-1)-curve exists up to $m\leq 6$. Hence a numerical classification of these systems gives a classification for all

quasi-homogeneous (-1)-curves up to multiplicity $m = 6$. Such a classification is given in [CM98].

Now we consider the following phenomenon: Let $\mathcal{L} = \mathcal{L}(d, m_0, m^n)$ be a quasi-homogeneous linear system and $\mathcal{A}$ a (-1)-curve such that $\mathcal{A} = \mathcal{L}(\delta, \mu_0, \mu_1, \ldots, \mu_n)$ and $\mathcal{L}.\mathcal{A} \leq -2$. Let Perm_n be the permutation group on n letters and let $\sigma \in \mathrm{Perm}_n$. We define $\mathcal{A}_\sigma = \mathcal{L}(\delta, \mu_0, \mu_{\sigma(1)}, \ldots, \mu_{\sigma(n)})$. Then, as $\mathcal{A}$ is a (-1)-curve, it follows that $\mathcal{A}_\sigma$ is again a (-1)-curve. As $\mathcal{L}$ is quasi-homogeneous we have again $\mathcal{L}.\mathcal{A}_\sigma \leq -2$. Therefore we can construct a composition of (-1)-curves, which split off the system $\mathcal{L}$. We define the set $A \subset \mathrm{Perm}_n$ to be maximal such that all $\mathcal{A}_\sigma$ with $\sigma \in A$ are pairwise different. Then we define a new plane curve $\mathcal{A}_{tot} = \sum_{\sigma \in A} \mathcal{A}_\sigma$ (see [LU02]).

We call a linear system $\mathcal{L}' = \mathcal{L}(d, m_o, m_1, \ldots, m_n)$ as above a *quasi-homogeneous (-1)-configuration* if $\mathcal{A}_{\mathrm{tot}}$ is a generic element in $\mathcal{L}'$. We note that $\mathcal{L}'$ is by construction quasi-homogeneous (if $k = |A|$ then there exists a μ' such that $\mathcal{L}' = \mathcal{L}(k\delta, k\mu_0, \mu'^n)$).

4.1 Lemma (splitting-off Lemma)
Let $\mathcal{L} = \mathcal{L}(d, m_0, m^n)$. Then every (-1)-curve $\mathcal{A}$ with $\mathcal{L}.\mathcal{A} \leq -2$ is of one of the following types (We have listed the associated quasi-homogeneous compound (-1)-configurations, too.):

$$\begin{array}{ll} \mathcal{A} = \mathcal{L}(\delta, \mu_0, \mu_1^n) & \\ \mathcal{A} = \mathcal{L}(\delta, \mu_0, \mu_2 - 1, \mu_2^{n-1}) & \mathcal{A}_{tot} = \mathcal{L}(n\delta, n\mu_0, (n\mu_2 - 1)^n) \\ \mathcal{A} = \mathcal{L}(\delta, \mu_0, \mu_2 + 1, \mu_2^{n-1}) & \mathcal{A}_{tot} = \mathcal{L}(n\delta, n\mu_0, (n\mu_2 + 1)^n) \end{array}$$

Proof:
First one proves that strict transforms of different $\mathcal{A}_\sigma \neq \mathcal{A}_{\sigma'}$ cannot meet positively on $\mathbb{P}'$. This is the case as otherwise one sees, by the Riemann-Roch theorem on $\mathbb{P}'$, that the sum of these moves in a linear system of positive dimension, which is a contradiction to being a fixed part of $\mathcal{L}$. This implies that all the different $\mathcal{A}_\sigma$ are linearly independent in $\mathrm{Pic}(\mathbb{P}')$. Let the $\mu_1, \ldots, \mu_n$ occur in sets of size $k_1 \leq \ldots \leq k_s$. As $\mathrm{rank}\,\mathrm{Pic}(\mathbb{P}') = n + 2$ we see by combinatorial reasons that for the $\frac{n!}{k_1! \cdots k_s!}$ different (-1)-curves $\mathcal{A}_\sigma$ only the possibilities

$$\begin{array}{ll} s = 1,\ k_1 = n & \text{or} \\ s = 2,\ k_1 = 1,\ k_2 = n - 1 & \end{array}$$

can occur. That means we have at most three different multiplicities μ_0, μ_1 and μ_2.

Moreover we have the equations $\mathcal{A}.\mathcal{A} = -1$ and $\mathcal{A}.\mathcal{A}_\sigma = 0$ on $\mathbb{P}'$. That gives $\mathcal{A}.\mathcal{A} - \mathcal{A}.\mathcal{A}_\sigma = -1$ which is equivalent to $(\mu_1 - \mu_2)^2 = 1$ (see [CM98]). □

For the purpose of classifying the systems $\mathcal{L}(d, m_0, 6^n)$ we need a complete list of all (-1)-curves which might split off such systems two times. These (-1)-curves can not have higher multiplicities than 3 at the points $p_1, \ldots, p_n$. We obtain the following result:

4.2 Lemma (classification of (-1)-curves)
All (-1)-curves $\mathcal{A}$ and quasi-homogeneous (-1)-configurations $\mathcal{A}_{\mathrm{tot}}$ up to multiplicity 3 in the points $p_1, \ldots, p_n$ which might split off a quasi-homogeneous system $\mathcal{L} = \mathcal{L}(d, m_0, 6^n)$ are elements of the systems in the following list (see [LU02]*):*

not compound		*compound*	
$\mathcal{L}(2,0,1^5)$			
$\mathcal{L}(e,e-1,1^{2e})$	$e \geq 1$		
$\mathcal{L}(1,1,1^1)$		$\mathcal{L}(n,n,1^n)$	$n \geq 2$
$\mathcal{L}(1,0,1^2)$		$\mathcal{L}(3,0,2^3)$	
$\mathcal{L}(6,3,2^7)$			
$\mathcal{L}(12,8,3^9)$			

In particular, all the (-1)-curves are quasi-homogeneous.

Proof:
We refer to [CM98, Example 5.1] for the proof of a list of all quasi-homogeneous (-1)-classes up to multiplicity 3. In [CM98, Example 5.15] is given a complete list of all quasi-homogeneous (-1)-configurations up to multiplicity 3. Using this two lists and Lemma 4.1 gives this result. □

Now we give the proof of the classification theorem of all (-1)-special systems of the form $\mathcal{L}(d, m_0, 6^n)$.

Proof of Theorem B:

In lemma 4.2 we have seen the possible cases for (-1)-curves which might split off $\mathcal{L}(d, m_0, 6^n)$. Now we have to consider all these cases. To be a little bit faster we proceed along the following algorithm (see [LU02]):

We go through all possible combinations of these (-1)-curves step by step.

First step: If we find a (-1)-curve or a (-1)-configuration $\mathcal{A}$ such that

$$\mathcal{L}.\mathcal{A} = -\mu \leq -2,$$

then we split off the fixed part and define $\mathcal{M} = \mathcal{L} - \mu \cdot \mathcal{A}$.

Second step: Let $\mathcal{M}'$ be the residual system of $\mathcal{M}$ obtained by splitting off all possible (-1)-curves. By the definition of (-1)-special systems we have to verify that $v(\mathcal{M}') \geq 0$. We notice that the systems $\mathcal{M}$ are quasi-homogeneous of multiplicity ≤ 4 by lemma 4.2. Therefore we can use the results of [CM98] and [S99].

- $\boxed{\mathcal{L} = \mathcal{M} + \mu \cdot \mathcal{A}, \ \mathbf{v}(\mathcal{M}) \geq \mathbf{0} \text{ and } \mathcal{M}.\mathcal{A} = \mathbf{0}}$
  (1) $\mathcal{A} = \mathcal{L}(\mathbf{2}, \mathbf{0}, \mathbf{1^5})$ **and** $\mathcal{L} = \mathcal{L}(\mathbf{d}, \mathbf{m_0}, \mathbf{6^5})$

  This gives $\mathcal{M} = \mathcal{L}(d-2n, m_0, (6-\mu)^5)$ and $\mathcal{M}.\mathcal{A} = 0$ gives $d = \frac{30-\mu}{2}$.

  If $\underline{\mu = 2} \Longrightarrow d = 14$ and we get
  $m_0 = 0$ and $v(\mathcal{M}) = 15$ with $\mathcal{M} = \mathcal{L}(10, 0, 4^5)$, non-special by [S99]
  $m_0 = 1$ and $v(\mathcal{M}) = 14$ with $\mathcal{M} = \mathcal{L}(10, 1, 4^5)$, ″
  $m_0 = 2$ and $v(\mathcal{M}) = 12$ with $\mathcal{M} = \mathcal{L}(10, 2, 4^5)$, ″
  $m_0 = 3$ and $v(\mathcal{M}) = 9$ with $\mathcal{M} = \mathcal{L}(10, 3, 4^5)$, ″
  $m_0 = 4$ and $v(\mathcal{M}) = 5$ with $\mathcal{M} = \mathcal{L}(10, 4, 4^5)$, ″
  $m_0 = 5$ and $v(\mathcal{M}) = 0$ with $\mathcal{M} = \mathcal{L}(10, 5, 4^5)$, ″

  $\underline{\mu = 3}$ is not possible because of $\mathcal{M}.\mathcal{A} = 0$.

  If $\underline{\mu = 4} \Longrightarrow d = 13$ and we conclude
  $m_0 = 0$ and $v(\mathcal{M}) = 5$ with $\mathcal{M} = \mathcal{L}(7, 0, 2^5)$, non-special by [CM98]
  $m_0 = 1$ and $v(\mathcal{M}) = 4$ with $\mathcal{M} = \mathcal{L}(7, 1, 2^5)$, ″
  $m_0 = 2$ and $v(\mathcal{M}) = 2$ with $\mathcal{M} = \mathcal{L}(7, 2, 2^5)$, ″
  $m_0 = 3$ and $v(\mathcal{M}) = -1$

$\underline{\mu = 5}$ is not possible because of $\mathcal{M}.\mathcal{A} = 0$.

From $\underline{\mu = 6} \Longrightarrow d = 12$ and $m_0 = 0$, $v(\mathcal{M}) = 0$ for $\mathcal{M} = \mathcal{L}(0,0)$.

(2) $\mathbf{\mathcal{A} = \mathcal{L}(e, e-1, 1^{2e})}$ **$\mathbf{e \geq 1}$ and $\mathbf{\mathcal{L} = \mathcal{L}(d, m_0, 6^{2e})}$**

Then follows $\mathcal{M} = \mathcal{L}(d - \mu \cdot e, m_0 - \mu \cdot e + \mu, (6-\mu)^{2e})$ and $\mathcal{M}.\mathcal{A} = 0$ gives $-e \cdot m_0 + e \cdot d - 12e + m_0 + \mu = 0 \Longrightarrow m_0 > d - 12$. $v(\mathcal{M}) \geq 0$ gives $d \geq m_0 + \mu - 2$.

$\underline{\mu = 2} \Longrightarrow d \geq m_0 > d - 12$

m_0	$v(\mathcal{M}) \leq -1$ and $\mathcal{M}$ non-special
d	"
$d-1$	"

m_0	from $\mathcal{M}.\mathcal{A} = 0 \Rightarrow d$	residual system
$d-2$	$10e$	$\mathcal{M} = \mathcal{L}(8e, 8e, 4^{2e})$ irregular by [S99] $\Rightarrow$ non-empty
$d-3$	$9e+1$	$\mathcal{M} = \mathcal{L}(7e+1, 7e, 4^{2e})$ irregular by [S99] $\Rightarrow$ non-empty
$d-4$	$8e+2$	$\mathcal{M} = \mathcal{L}(6e+2, 6e, 4^{2e})$ irregular by [S99] $\Rightarrow$ non-empty
$d-5$	$7e+3$	$\mathcal{M} = \mathcal{L}(5e+3, 5e, 4^{2e})$ regular by [S99] and $v(\mathcal{M}) = 9$
$d-6$	$6e+4$	$\mathcal{M} = \mathcal{L}(4e+4, 4e, 4^{2e})$ regular by [S99] and $v(\mathcal{M}) = 14$
$d-7$	$5e+5$	$\mathcal{M} = \mathcal{L}(3e+5, 3e, 4^{2e})$ regular by [S99] and $v(\mathcal{M}) = -2e+20$
$d-8$	$4e+6$	$\mathcal{M} = \mathcal{L}(2e+6, 2e, 4^{2e})$ regular by [S99] and $v(\mathcal{M}) = -6e+27$
$d-9$	$3e+7$	$\mathcal{M} = \mathcal{L}(e+7, e, 4^{2e})$ regular by [S99] and $v(\mathcal{M}) = -12e+35$
$d-10$	$2e+8$	$\mathcal{M} = \mathcal{L}(8, 0, 4^{2e})$ regular by [S99] and $v(\mathcal{M}) = -20e+44$
$d-11$	$e+9$	$\Rightarrow m_0 \leq -1$ not possible

$\underline{\mu = 3} \Longrightarrow d - 1 \geq m_0 > d - 12$

m_0	$v(\mathcal{M}) \leq -1$ and $\mathcal{M}$ non-special
$d-1$	"
$d-2$	"

m_0	from $\mathcal{M}.\mathcal{A} = 0 \Rightarrow d$	residual system
$d-3$	$9e$	$\mathcal{M} = \mathcal{L}(6e, 6e, 3^{2e})$ irregular by [CM98] and $e(\mathcal{M}) = 0$
$d-4$	$8e+1$	$\mathcal{M} = \mathcal{L}(5e+1, 5e, 3^{2e})$ irregular by [CM98] and $e(\mathcal{M}) = 2$
$d-5$	$7e+2$	$\mathcal{M} = \mathcal{L}(4e+2, 4e, 3^{2e})$ regular by [CM98] and $v(\mathcal{M}) = 5$
$d-6$	$6e+3$	$\mathcal{M} = \mathcal{L}(3e+3, 3e, 3^{2e})$ regular by [CM98] and $v(\mathcal{M}) = 9$

m_0	from $\mathcal{M}.\mathcal{A}=0 \Rightarrow d$	residual system
$d-7$	$5e+4$	$\mathcal{M}=\mathcal{L}(2e+4,2e,3^{2e})$ regular by [CM98] and $v(\mathcal{M})=-2e+14$
$d-8$	$4e+5$	$\mathcal{M}=\mathcal{L}(e+5,e,3^{2e})$ regular by [CM98] and $v(\mathcal{M})=-6e+20$
$d-9$	$3e+6$	$\mathcal{M}=\mathcal{L}(6,0,3^{2e})$ regular by [CM98] and $v(\mathcal{M})=-12e+27$
$d-10$	$2e+7$	$\Rightarrow m_0 \leq -1$ not possible

$\underline{\mu=4} \Longrightarrow d-2 \geq m_0 > d-12$

m_0	from $\mathcal{M}.\mathcal{A}=0 \Rightarrow d$	residual system
$d-2$	$10e-2$	$\mathcal{M}=\mathcal{L}(6e-2,6e,2^{2e})$ empty
$d-3$	$9e-1$	$\mathcal{M}=\mathcal{L}(5e-1,5e,2^{2e})$ empty
$d-4$	$8e$	$\mathcal{M}=\mathcal{L}(4e,4e,2^{2e})$ irregular by [CM98] and $e(\mathcal{M})=0$
$d-5$	$7e+1$	$\mathcal{M}=\mathcal{L}(3e+1,3e,2^{2e})$ regular by [CM98] and $v(\mathcal{M})=2$
$d-6$	$6e+2$	$\mathcal{M}=\mathcal{L}(2e+2,2e,2^{2e})$ regular by [CM98] and $v(\mathcal{M})=5$
$d-7$	$5e+3$	$\mathcal{M}=\mathcal{L}(e+3,e,2^{2e})$ regular by [CM98] and $v(\mathcal{M})=-2e+9$
$d-8$	$4e+4$	$\mathcal{M}=\mathcal{L}(4,0,2^{2e})$ regular by [CM98] and $v(\mathcal{M})=-6e+14$
$d-9$	$3e+5$	$\Rightarrow m_0 \leq -1$ not possible

$\underline{\mu=5} \Longrightarrow d-3 \geq m_0 > d-12$

m_0	from $\mathcal{M}.\mathcal{A}=0 \Rightarrow d$	residual system
$d-3$	$9e-2$	$\mathcal{M}=\mathcal{L}(4e-2,4e,1^{2e})$ empty
$d-4$	$8e-1$	$\mathcal{M}=\mathcal{L}(3e-1,3e,1^{2e})$ empty
$d-5$	$7e$	$\mathcal{M}=\mathcal{L}(2e,2e,1^{2e})$ regular by [CM98] and $v(\mathcal{M})=0$
$d-6$	$6e+1$	$\mathcal{M}=\mathcal{L}(e+1,e,1^{2e})$ regular by [CM98] and $v(\mathcal{M})=2$
$d-7$	$5e+2$	$\mathcal{M}=\mathcal{L}(2,0,1^{2e})$ regular by [CM98] and $v(\mathcal{M})=-2e+5$
$d-8$	$4e+3$	$\Rightarrow m_0 \leq -1$ not possible

For $\underline{\mu=6}$ we have that $d-4 \geq m_0 > d-12$. Let $m_0=d-x$. From $\mathcal{M}.\mathcal{A}=0$ $\Rightarrow d=(12-x)e+(x-6)$. We notice that $\mathcal{M}=\mathcal{L}((6-x)e+(x-6),(6-x)e,0)$, which is regular. Taking into account that $v(\mathcal{M}) \leq -1$ for all $x \leq 5$ and $m_0 \leq -1$ for all $x \geq 7$ we get the only case:
$m_0=d-6$ and $\mathcal{M}.\mathcal{A}=0 \Rightarrow d=6e$ and $\mathcal{M}=\mathcal{L}(0,0)$ is regular with $v(\mathcal{M})=0$.

(3) $\mathcal{A}=\mathcal{L}(\mathbf{e},\mathbf{e},\mathbf{1^e})$ **and** $\mathcal{L}=\mathcal{L}(\mathbf{d},\mathbf{m_0},\mathbf{6^e})$

This leads to $\mathcal{M}=\mathcal{L}(d-\mu e,m_0-\mu e,(6-\mu)^e)$. $\mathcal{M}.\mathcal{A}=0$ gives $m_0=d+\mu-6$. If $\underline{\mu=2}$ then we get $m_0=d-4$, $\mathcal{L}=\mathcal{L}(d,d-4,6^e)$ and $\mathcal{M}=\mathcal{L}(d-2e,d-4-2e,4^e)$. From $v(\mathcal{M})=-20e+5d-6 \Longrightarrow v(\mathcal{M}) \geq 0$ if $d \geq \frac{6+20e}{5}$. Furthermore $\mathcal{M}$ is irregular by [S99] and of higher dimension if
(1) $e=2f$ and $d=8f$
(2) $e=2f$ and $d=8f+1$
(3) $e=2f$ and $d=8f+2$.

If $\underline{\mu = 3}$ then we get $m_0 = d-3$, $\mathcal{L} = \mathcal{L}(d, d-3, 6^e)$ and $\mathcal{M} = \mathcal{L}(d-3e, d-3-3e, 3^e)$. From $v(\mathcal{M}) = -18e+4d-3 \Longrightarrow v(\mathcal{M}) \geq 0$ if $d \geq \frac{3+18e}{4}$. Further $\mathcal{M}$ is irregular by [CM98] and of higher dimension if

(1) $e = 2f$, $d = 9f$ and $e(\mathcal{M}) = 0$ or
(2) $e = 2f$, $d = 9f+1$ and $e(\mathcal{M}) = 2$.

If $\underline{\mu = 4}$ then $m_0 = d-2$, $\mathcal{L} = \mathcal{L}(d, d-2, 6^e)$ and $\mathcal{M} = \mathcal{L}(d-4e, d-2-4e, 2^e)$. From $v(\mathcal{M}) = -15e+3d-1 \Longrightarrow v(\mathcal{M}) \geq 0$ if $d \geq \frac{1+15e}{3}$. Further $\mathcal{M}$ is irregular by [CM98] and of higher dimension if $e = 2f$, $d = 10f$ and $e(\mathcal{M}) = 0$.

If $\underline{\mu = 5}$ then $m_0 = d-1$, $\mathcal{L} = \mathcal{L}(d, d-1, 6^e)$ and $\mathcal{M} = \mathcal{L}(d-5e, d-1-5e, 1^e)$. From $v(\mathcal{M}) = -11e+2d \Longrightarrow v(\mathcal{M}) \geq 0$ if $d \geq \frac{11e}{2}$. $\mathcal{M}$ is always regular by [CM98].

If $\underline{\mu = 6}$ then $m_0 = d$, $\mathcal{L} = \mathcal{L}(d, d, 6^e)$ and $\mathcal{M} = \mathcal{L}(d-6e, d-6e)$. $v(\mathcal{M}) = -6e+d \Longrightarrow v(\mathcal{M}) \geq 0$ if $d \geq 6e$. $\mathcal{M}$ is always regular.

The following two cases are easier to compute because we have no further parameters in the *(-1)-curves.*

(4) $\mathcal{A} = \mathcal{L}(\mathbf{6}, \mathbf{3}, \mathbf{2^7})$ **and** $\mathcal{L} = \mathcal{L}(\mathbf{d}, \mathbf{m_0}, \mathbf{6^7})$

This leads to $\mathcal{M} = \mathcal{L}(d-6\mu, m_0-3\mu, (6-2\mu)^7)$. $\mathcal{M}.\mathcal{A} = 0$ gives $m_0 = \frac{6d+\mu-84}{3}$. Therefore $\underline{\mu = 3}$ is the only possible case: $\mathcal{M} = \mathcal{L}(d-18, 2d-36)$. To get $v(\mathcal{M}) \geq 0$ we need $d = 18$. $\Longrightarrow \mathcal{L} = \mathcal{L}(18, 9, 6^7)$.

(5) $\mathcal{A} = \mathcal{L}(\mathbf{3}, \mathbf{0}, \mathbf{2^3})$ **and** $\mathcal{L} = \mathcal{L}(\mathbf{d}, \mathbf{m_0}, \mathbf{6^3})$

This leads to $\mathcal{M} = \mathcal{L}(d-3\mu, m_0, (6-2\mu)^3)$. $\mathcal{M}.\mathcal{A} = 0$ gives $d = 12-\mu$.

$\underline{\mu = 2}$ We get $\mathcal{L} = \mathcal{L}(10, m_0, 2^3)$ and $\mathcal{M} = \mathcal{L}(4, m_0, 2^3)$. From $v(\mathcal{M}) \geq 0$ we get $m_0 \in \{0, 1, 2\}$. All $\mathcal{M}$ are regular by [CM98].

$\underline{\mu = 3}$ We get $\mathcal{L} = \mathcal{L}(9, m_0)$ and $\mathcal{M} = \mathcal{L}(0, m_0)$. $\Longrightarrow m_0 = 0$ and $v(\mathcal{M}) = 0$.

(6) $\mathcal{A} = \mathcal{L}(\mathbf{12}, \mathbf{8}, \mathbf{3^9})$**,** $\mathcal{L} = \mathcal{L}(\mathbf{d}, \mathbf{m_0}, \mathbf{6^9})$ **and** $\mu = \mathbf{2}$

This lead to $\Longrightarrow \mathcal{M} = \mathcal{L}(d-24, m_0-16)$, which is regular. From $\mathcal{M}.\mathcal{A} = 0$ we get $m_0 = \frac{3d-40}{2}$. Therefore $v(\mathcal{M}) \geq 0$ gives $d \in \{24, 25\}$, but only $d = 24$ and $m_0 = 16$ is possible.

$$\boxed{\mathcal{L} = \mathcal{M} + \mathbf{2} \cdot \mathcal{A}_\mathbf{1} + \mathbf{2} \cdot \mathcal{A}_\mathbf{2}\textbf{, } \mathbf{v}(\mathcal{M}) \geq \mathbf{0}\textbf{, } \mathcal{M} \textbf{ non-special and } \mathcal{M}.\mathcal{A} = \mathbf{0}}$$

(1) $\mathcal{A} = \mathcal{L}(\delta, \mu_\mathbf{0}, \mathbf{1^n})$ **and** $\mathcal{A}_\mathbf{1}.\mathcal{A}_\mathbf{2} = \mathbf{0}$

This leads to $\mathcal{A}_1 = \mathcal{L}(e, e-1, 1^{2e})$ and $\mathcal{A}_2 = \mathcal{L}(2e, 2e, 1^{2e})$. Further we have $\mathcal{L} = \mathcal{L}(d, m_0, 6^{2e})$ and $\mathcal{M} = \mathcal{L}(d-6e, m_0-6e+2, 2^{2e})$. From $\mathcal{M}.\mathcal{A}_1 = 0$ and $\mathcal{M}.\mathcal{A}_2 = 0$ we get $m_0 = d-4$ and $d = 8e+2$. Therefore we have $\mathcal{M} = \mathcal{L}(2e+2, 2e, 2^{2e})$, which is regular by [CM98] and $v(\mathcal{M}) = 5$.

(2) $\mathcal{A}_\mathbf{1} = \mathcal{L}(\delta_\mathbf{1}, \mu_{\mathbf{0_1}}, \mathbf{1^n})$ **and** $\mathcal{A}_\mathbf{2} = \mathcal{L}(\delta_\mathbf{2}, \mu_{\mathbf{0_2}}, \mathbf{2^n})$

$\mathcal{A}_1.\mathcal{A}_2 = 0$ gives only the following possibilities:

(1) $\mathcal{A}_1 \in \mathcal{L}(2, 1, 1^4)$ and $\mathcal{A}_2 \in \mathcal{L}(3, 0, 2^3)$,
(2) $\mathcal{A}_1 \in \mathcal{L}(2, 0, 1^5)$ and $\mathcal{A}_2 \in \mathcal{L}(3, 0, 2^3)$ or
(3) $\mathcal{A}_1 \in \mathcal{L}(2, 2, 1^2)$ and $\mathcal{A}_2 \in \mathcal{L}(3, 0, 2^3)$.

(1) and (2) are not possible cases as these curves are elements of quasi-homogeneous systems based on a different number of points with equal multiplicities. It is not possible to find a suitable system $\mathcal{L}(d, m_0, 6^n)$. So let us focus on (3), where we see that it is equivalent to assume $\mathcal{A}_2 \in \mathcal{L}(3, 2, 2^2)$. In this case we see that $\mathcal{L} = \mathcal{L}(d, m_0, 6^2)$ and $\mathcal{M} = \mathcal{L}(d-10, m_0-8)$. From $\mathcal{M}.\mathcal{A}_1 = 0$ and $\mathcal{M}.\mathcal{A}_2 = 0$ we conclude that $d = 10$ and $m_0 = 8$, that means we get the system $\mathcal{L} = \mathcal{L}(10, 8, 6^2)$.

$\boxed{\mathcal{L} = \mathcal{M} + \mathbf{2} \cdot \mathcal{A}_\mathbf{1} + \mathbf{3} \cdot \mathcal{A}_\mathbf{2}, \ \mathbf{v}(\mathcal{M}) \geq \mathbf{0} \text{ and } \mathcal{M}.\mathcal{A} = \mathbf{0}}$

$\boxed{\mathcal{A} = \mathcal{L}(\delta, \mu_\mathbf{0}, \mathbf{1^n}) \text{ and } \mathcal{A}_\mathbf{1}.\mathcal{A}_\mathbf{2} = \mathbf{0}}$

(1) $\mathcal{A}_\mathbf{1} = \mathcal{L}(\mathbf{e}, \mathbf{e}-\mathbf{1}, \mathbf{1^{2e}})$ & $\mathcal{A}_\mathbf{2} = \mathcal{L}(\mathbf{2e}, \mathbf{2e}, \mathbf{1^{2e}})$

Moreover we have $\mathcal{L} = \mathcal{L}(d, m_0, 6^{2e})$ and $\mathcal{M} = \mathcal{L}(d-8e, m_0-8e+2, 1^{2e})$. From $\mathcal{M}.\mathcal{A}_1 = 0$ and $\mathcal{M}.\mathcal{A}_2 = 0$ we get $m_0 = d-3$ and $d = 9e+1$. Therefore we have $\mathcal{M} = \mathcal{L}(e+1, e, 1^{2e})$ which is regular by [CM98] and $v(\mathcal{M}) = 2$.

(2) $\mathcal{A}_\mathbf{1} = \mathcal{L}(\mathbf{2e}, \mathbf{2e}, \mathbf{1^{2e}})$ & $\mathcal{A}_\mathbf{2} = \mathcal{L}(\mathbf{e}, \mathbf{e}-\mathbf{1}, \mathbf{1^{2e}})$

Furthermore we have $\mathcal{L} = \mathcal{L}(d, m_0, 6^{2e})$ and $\mathcal{M} = \mathcal{L}(d-7e, m_0-7e+3, 1^{2e})$. From $\mathcal{M}.\mathcal{A}_1 = 0$ and $\mathcal{M}.\mathcal{A}_2 = 0$ we get $m_0 = d-4$ and $d = 8e+1$. Therefore we have $\mathcal{M} = \mathcal{L}(e+1, e, 1^{2e})$ which is regular by [CM98] and $v(\mathcal{M}) = 2$.

$\boxed{\mathcal{L} = \mathcal{M} + \mathbf{2} \cdot \mathcal{A}_\mathbf{1} + \mathbf{4} \cdot \mathcal{A}_\mathbf{2}, \ \mathbf{v}(\mathcal{M}) \geq \mathbf{0} \text{ and } \mathcal{M}.\mathcal{A} = \mathbf{0}}$

$\boxed{\mathcal{A} = \mathcal{L}(\delta, \mu_\mathbf{0}, \mathbf{1^n}) \text{ and } \mathcal{A}_\mathbf{1}.\mathcal{A}_\mathbf{2} = \mathbf{0}}$

(1) $\mathcal{A}_\mathbf{1} = \mathcal{L}(\mathbf{e}, \mathbf{e}-\mathbf{1}, \mathbf{1^{2e}})$ **and** $\mathcal{A}_\mathbf{2} = \mathcal{L}(\mathbf{2e}, \mathbf{2e}, \mathbf{1^{2e}})$

Moreover we have $\mathcal{L} = \mathcal{L}(d, m_0, 6^{2e})$ and $\mathcal{M} = \mathcal{L}(d-10e, m_0-10e+2)$. From $\mathcal{M}.\mathcal{A}_1 = 0$ and $\mathcal{M}.\mathcal{A}_2 = 0$ we get $m_0 = d-2$ and $d = 10e$. Therefore we get $\mathcal{M} = \mathcal{L}(0, 0)$ and $v(\mathcal{M}) = 0$.

(2) $\mathcal{A}_\mathbf{1} = \mathcal{L}(\mathbf{2e}, \mathbf{2e}, \mathbf{1^{2e}})$ **and** $\mathcal{A}_\mathbf{2} = \mathcal{L}(\mathbf{e}, \mathbf{e}-\mathbf{1}, \mathbf{1^{2e}})$

Furthermore we have $\mathcal{L} = \mathcal{L}(d, m_0, 6^{2e})$ and $\mathcal{M} = \mathcal{L}(d-8e, m_0-8e+4)$. From $\mathcal{M}.\mathcal{A}_1 = 0$ and $\mathcal{M}.\mathcal{A}_2 = 0$ we get $m_0 = d-4$ and $d = 8e$. Therefore we have $\mathcal{M} = \mathcal{L}(0, 0)$ and $v(\mathcal{M}) = 0$.

$\boxed{\mathcal{L} = \mathcal{M} + \mathbf{2} \cdot \mathcal{A}_\mathbf{1} + \mathbf{2} \cdot \mathcal{A}_\mathbf{2} + \mathbf{2} \cdot \mathcal{A}_\mathbf{3}, \ \mathbf{v}(\mathcal{M}) \geq \mathbf{0} \text{ and } \mathcal{M}.\mathcal{A} = \mathbf{0}}$

As $\mathcal{A}_1 = \mathcal{L}(e, e-1, 1^{2e})$ and $\mathcal{A}_2 = \mathcal{L}(e, e, 1^e)$ are the only compound (-1)-configurations with multiplicity $m = 1$ in $p_1, \ldots, p_n$ which have intersection multiplicity $= 0$. Therefore we are immediately in case 4.

This finally completes our proof of the classification theorem. $\square$

5. The Degeneration Method

In this section we give a rough overview of the degeneration of the plane as introduced by Ciliberto and Miranda in [CM98]. We refer to this paper for further details. As in every degeneration method the aim is to specialize the base points of a system $\mathcal{L}(d, m_0, m^n)$ in such a way that on the one hand the dimension is easier to compute but on the other hand it does not change.

At first we consider the geometric situation. Let Δ be a complex disc around the origin. We define $V = \mathbb{P}^2 \times \Delta$. Let $p_1 : V \longrightarrow \mathbb{P}^2$ and $p_2 : V \longrightarrow \Delta$ be the projections. Now we blow up a line L in $V_0 = p_2^{-1}(0)$ ($f : X \longrightarrow V$) and obtain the following situation with $\pi_i = f \circ p_i$:

$$\begin{tikzcd}
 & X \arrow[d, "f"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & V \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
\mathbb{P}^2 & & \Delta
\end{tikzcd}
\qquad X_0 = \mathbb{P} \cup_{\mathcal{R}} \mathbb{F}
\qquad
\begin{tikzcd}
\mathbb{F} \arrow[dr, "\sigma"] & \\
 & \mathbb{P}^2
\end{tikzcd}$$

Now $X_t = \pi_2^{-1}(t) \cong \mathbb{P}^2$ for all $t \neq 0$. $X_0 = \pi_2^{-1}(0)$ is a union of two surfaces, the strict transform of $V_0 \cong \mathbb{P}^2$ (called $\mathbb{P}$) and the exceptional divisor $\mathbb{F} = f^{-1}(L)$. $\mathbb{F}$ is isomorphic to the blow-up of $\mathbb{P}^2$ in one point p (here via σ). The surfaces are glued together along the line $\mathcal{R}$, which can be identified with L in $\mathbb{P}$ and with the exceptional divisor $E = \sigma^{-1}(p)$ in $\mathbb{F}$.

As in [CM98] we define $\mathcal{O}_X(d) = \pi_1^* \mathcal{O}_{\mathbb{P}^2}(d)$ and $\mathcal{O}_X(d,k) = \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(k\mathbb{P})$. We set $\chi(d,k) = \mathcal{O}_X(d,k)|_{X_0}$. Let H be the pull-back of a general line in $\mathbb{P}^2$ via σ. Then we have $\mathcal{O}_X(d,k)|_{X_t} \cong \mathcal{O}_{\mathbb{P}^2}(d)$ for $t \neq 0$. Furthermore $\chi(d,k)|_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}^2}(d-k)$ and $\chi(d,k)|_{\mathbb{F}} \cong \mathcal{O}_{\mathbb{F}}(dH - (d-k)E)$.

We fix $n - b + 1$ general points $p_0, p_1, \ldots, p_{n-b}$ on $\mathbb{P}$ and b general points $p_{n-b+1}, \ldots, p_n$ on $\mathbb{F}$. We define $\mathcal{L}_0$ to be the linear sub-system of $\chi(d,k)$ defined by all divisors of $\chi(d,k)$ having multiplicity at least m_0 at p_0 and at least m at the points $p_1, \ldots, p_n$ (write $\mathcal{L}_0 = \mathcal{L}(d, m_0, m^{n-b}, m^b))$. We say that $\mathcal{L}_0$ is obtained from $\mathcal{L} = \mathcal{L}(d, m_0, m^n)$ by an *(k,b)-degeneration*. $\mathcal{L}_0$ can be considered as a flat limit on X_0 of $\mathcal{L}$. By semi-continuity we obtain

$$\ell_0 = \ell(\mathcal{L}_0) \geq \ell(\mathcal{L}).$$

In particular, if $\ell_0 = e(\mathcal{L})$ then $\mathcal{L}$ is non-special.

Now $\mathcal{L}_0$ restricts on $\mathbb{P}$ to a system $\mathcal{L}_{\mathbb{P}} = \mathcal{L}(d-k, m_0, m^{n-b})$. Furthermore we restrict $\mathcal{L}_0$ on $\mathbb{F}$ to $\mathcal{L}_{\mathbb{F}} = \mathcal{L}(d, d-k, m^b)$ (the identification we obtain by blowing down $\mathcal{L}_{\mathbb{F}}$ to $\mathbb{P}^2$ via σ). Now we define as in [CM98] $\mathcal{R}_{\mathbb{P}}$ to be the linear system on $\mathcal{R}$ obtained by restricting $\mathcal{L}_{\mathbb{P}}$ to $\mathcal{R}$. We have the following exact sequence

$$0 \longrightarrow \hat{\mathcal{L}}_{\mathbb{P}} \xrightarrow{+L} \mathcal{L}_{\mathbb{P}} \xrightarrow{|_L} \mathcal{R}_{\mathbb{P}} \longrightarrow 0.$$

The kernel system $\hat{\mathcal{L}}_{\mathbb{P}}$ consists of all divisors having L as component. So we can identify $\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(d-k-1, m_0, m^{n-b})$.

We analogously define $\mathcal{R}_{\mathbb{F}}$ and obtain $\hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(d, d-k+1, m^b)$ (parametrising the divisors in $\mathcal{L}_{\mathbb{F}}$ which have E as a component).

Let us recall some further abbreviations from [CM98]:

5.1 Definitions

$v_{\mathbb{P}} = v(\mathcal{L}_{\mathbb{P}}),\ v_{\mathbb{F}} = v(\mathcal{L}_{\mathbb{F}}),$
$\hat{v}_{\mathbb{P}} = v(\hat{\mathcal{L}}_{\mathbb{P}}),\ \hat{v}_{\mathbb{F}} = v(\hat{\mathcal{L}}_{\mathbb{F}}),$
$\ell_{\mathbb{P}} = \ell(\mathcal{L}_{\mathbb{P}}),\ \ell_{\mathbb{F}} = \ell(\mathcal{L}_{\mathbb{F}}),$
$\hat{\ell}_{\mathbb{P}} = \ell(\hat{\mathcal{L}}_{\mathbb{P}}),\ \hat{\ell}_{\mathbb{F}} = \ell(\hat{\mathcal{L}}_{\mathbb{F}}),$
$r_{\mathbb{P}} = \ell_{\mathbb{P}} - \hat{\ell}_{\mathbb{P}} - 1$, *the dimension of* $\mathcal{R}_{\mathbb{P}}$,
$r_{\mathbb{F}} = \ell_{\mathbb{F}} - \hat{\ell}_{\mathbb{F}} - 1$, *the dimension of* $\mathcal{R}_{\mathbb{F}}$.

In [CM98] it is shown that the associated vector spaces to $\mathcal{R}_{\mathbb{P}}$ and $\mathcal{R}_{\mathbb{F}}$ are transversal subspaces of $\Gamma(\mathcal{R}, \mathcal{O}_{\mathcal{R}}(d-k))$. This leads to the following corollary:

5.2 Corollary (Key-Lemma on ℓ_0)
We have two cases:

(1) *If* $r_{\mathbb{P}} + r_{\mathbb{F}} \leq d - k - 1$, *then* $\ell_0 \ = \hat{\ell}_{\mathbb{P}} + \hat{\ell}_{\mathbb{F}} + 1$.
(2) *If* $r_{\mathbb{P}} + r_{\mathbb{F}} \geq d - k - 1$, *then* $\ell_0 \ = \ell_{\mathbb{P}} + \ell_{\mathbb{F}} - d + k$.

A proof can be found in [CM98]

6. Proof of the Main Theorem

Before giving the proof let us state two lemmas which are corollaries of the Key-Lemma 5.2. The proof of these is given for an analogous case in [LU02].

6.1 Lemma (case $v(\mathcal{L}) \leq -1$)
Let $\mathcal{L} = \mathcal{L}(d, m_0, 6^n)$ *with* $v(\mathcal{L}) \leq -1$. *If there are integers* k $(k < d)$ *and* b $(b < n)$ *such that a* (k, b)*-degeneration can be found with the following properties of the restrictions of* $\mathcal{L}_0$

- $\mathcal{L}_{\mathbb{F}}$ *and* $\mathcal{L}_{\mathbb{P}}$ *are both non-special, and*
- *the kernel systems* $\hat{\mathcal{L}}_{\mathbb{F}}$ *and* $\hat{\mathcal{L}}_{\mathbb{P}}$ *are empty with* $\hat{v}_{\mathbb{P}} \leq v(\mathcal{L})$,

then $\mathcal{L}$ *is empty.*

6.2 Lemma (case $v(\mathcal{L}) \geq -1$)
Let $\mathcal{L} = \mathcal{L}(d, m_0, 6^n)$ *with* $v(\mathcal{L}) \geq -1$. *If there are integers* k $(k < d)$ *and* b $(b < n)$ *such that a* (k, b)*-degeneration can be found with*

- $\mathcal{L}_{\mathbb{F}}$ *and* $\mathcal{L}_{\mathbb{P}}$ *are both non-special,* $v_{\mathbb{P}} \geq -1$, $v_{\mathbb{F}} \geq -1$, *and*
- *the kernel systems* $\hat{\mathcal{L}}_{\mathbb{F}}$ *and* $\hat{\mathcal{L}}_{\mathbb{P}}$ *have the property* $v(\mathcal{L}) - 1 \geq \hat{\ell}_{\mathbb{P}} + \hat{\ell}_{\mathbb{F}}$,

then $\mathcal{L}$ *is non-special.*

The following three lemmas state parts of the result of the Main Theorem A. We prove them independently later on.

6.3 Lemma (three base points)
A linear system $\mathcal{L}(d, m_0, m^n)$ *with at most three base points* ($n \leq 2$) *is special if and only if it is* (-1)*-special.*

6.4 Lemma (large multiplicities m_0 in p_0)
Let $d \geq 25$. *If* $m_0 \geq d - 9$ *then* $\mathcal{L}(d, m_0, 6^n)$ *is special if and only if it is* (-1)*-special.*

6.5 Lemma (low degrees)
If $d \leq 140$ *then* $\mathcal{L}(d, m_0, 6^n)$ *is special if and only if it is* (-1)*-special.*

Proof of the Main Theorem A:

Let $\mathcal{L} = \mathcal{L}(d, m_0, 6^n)$. By the lemma for large multiplicities (6.4) we can assume that $d \geq m_0 + 10 \geq 10$.

Furthermore by the lemma for low degrees (6.5) the statement is true for $d \leq 140$. We can assume $d \geq 141$. We continue by induction on d where 6.5 can be considered as the base of the induction.

As all such $\mathcal{L}$ are not (-1)-special we have to show that $\mathcal{L}$ is non-special. The method is to get the system $\mathcal{L}_0$ on the special fiber by a degeneration of $\mathcal{L}$. With Lemmas 6.1 and 6.2 we can prove the regularity of $\mathcal{L}$ if the restrictions of $\mathcal{L}_0$ to $\mathbb{P}$ and to $\mathbb{F}$ have certain properties. These properties can be achieved as the main conjecture holds for the systems on $\mathbb{P}$ by induction and for the ones on $\mathbb{F}$ by 6.4.

We perform now a $(5, b)$-degeneration on $\mathcal{L}$ and get the following systems on the special fiber:

$$\mathbb{P}: \quad \mathcal{L}_{\mathbb{P}} = \mathcal{L}(d-5, m_0, 6^{n-b}) \quad \mathbb{F}: \quad \mathcal{L}_{\mathbb{F}} = \mathcal{L}(d, d-5, 6^b)$$

$$\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(d-6, m_0, 6^{n-b}) \qquad \hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(d, d-4, 6^b)$$

Step 1 (case $v(\mathcal{L}) \leq -1$):

We want to apply Lemma 6.1 for the case $v(\mathcal{L}) \leq -1$.

First of all we need to have $\hat{\mathcal{L}}_{\mathbb{F}}$ empty. By the lemma for large multiplicities in m_0 (6.4) we have that $\hat{\mathcal{L}}_{\mathbb{F}}$ is non-special if it is non-(-1)-special. Therefore by our classification theorem B it is sufficient to choose $d < 4b$, i.e., $b > \frac{d}{4}$. Also we get $\hat{v}_{\mathbb{F}} \leq -1$, which means this system is empty.

Next let us find a sufficient condition to get $\hat{v}_{\mathbb{P}} \leq v(\mathcal{L})$. A computation gives $\hat{v}_{\mathbb{P}} - v(\mathcal{L}) = -6d + 21b + 9$, hence it is sufficient to have $-6d + 21b + 9 \leq 0$, that is $b \leq \frac{6d-9}{21}$.

Now we want to find sufficient conditions to have $\mathcal{L}_{\mathbb{F}}$ non-special. By 6.4 this is already the case if we find conditions for $\mathcal{L}_{\mathbb{F}}$ not to be (-1)-special. By Theorem B it is sufficient to force $d > \frac{7b}{2} + 3$, that is $b < \frac{2}{7}(d-3)$. As $\frac{2}{7}(d-3) \leq \frac{6d-9}{21}$, this new condition on b includes also $\hat{v}_{\mathbb{P}} \leq v(\mathcal{L})$.

In the next step we are searching for a sufficient condition to get $\mathcal{L}_{\mathbb{P}}$ non-special. By induction on d $\mathcal{L}_{\mathbb{P}} = \mathcal{L}(d-5, m_0, 6^{n-b})$ is special if and only if it is (-1)-special. By our list in Theorem B we notice that $\mathcal{L}_{\mathbb{P}}$ is non-(-1)-special if we choose $n-b$ odd as we have assumed that $d - m_0 \geq 10$ and $d \geq 141$.

In the last step we look for a sufficient condition on b to get $\hat{\mathcal{L}}_{\mathbb{P}}$ empty. Here we have to be more careful. When $d - m_0 \geq 11$ we get for the same reasons as in the case of $\mathcal{L}_{\mathbb{P}}$ that $\hat{\mathcal{L}}_{\mathbb{P}}$ is non-special if $n-b$ is odd. When $d - m_0 = 10$ then from Theorem B we know that $\hat{\ell}_{\mathbb{P}} = -20(n-b) + 5(d-6) - 6$ if $n-b$ is odd. That means we want this expression to be negative. From $\hat{\ell}_{\mathbb{P}} \leq -1 \iff b \leq \frac{1}{4}(7-d) + n$ we get a sufficient condition on b. As by assumption $v(\mathcal{L}) \leq -1$, we can conclude that $v(\mathcal{L}) = 11d - 21n - 45 \leq -1$. Therefore $n \geq \frac{11d-44}{21}$. That means we can formulate the above condition on b without n (using a lower bound on n) and get $b \leq \frac{1}{4}(7-d) + \frac{11d-44}{21} = -\frac{29}{84} + \frac{23d}{84}$.

Let us now reformulate all sufficient conditions (separated for the cases $d - m_0 = 10$ and $d - m_0 > 10$) in a compact form: If $d - m_0 > 10$ we find a b such that we can apply Lemma 6.1 if

$$\frac{2}{7}d - \frac{6}{7} - \frac{1}{4}d > 2 \iff d \geq 81.$$

If $d-m_0=10$ we find also a b to apply 6.1 if

$$-\frac{29}{84}+\frac{23}{84}d-\frac{1}{4}d>2 \Longleftrightarrow d\geq 99.$$

<u>Step 2 (case $v(\mathcal{L})\geq -1$):</u>

We want to use Lemma 6.2 for the case $v(\mathcal{L})\geq -1$. Still all notations are with respect to the above $(5,b)$-degeneration.

In a first step we want to find a sufficient condition on b to get $\mathcal{L}_{\mathbb{P}}$ non-special. Exactly as in step 1 we get by induction that $\mathcal{L}_{\mathbb{P}}$ is non-special if we choose b such that $n-b$ is odd, because we assume $d-m_0\geq 10$ and $d\geq 141$.

Next we want to find sufficient conditions on b to get the system $\mathcal{L}_{\mathbb{F}}$ non-special and $v_{\mathbb{F}}\geq -1$. By the lemma for large multiplicities (6.4) in m_0 we have again as above that $\mathcal{L}_{\mathbb{F}}$ is non-special if and only if it is non-(-1)-special. We conclude that we get $\mathcal{L}_{\mathbb{F}}$ non-special if we have $d>\frac{7}{2}b+3$, that is if $b<\frac{2}{7}d-\frac{6}{7}$, by Theorem B. As $v_{\mathbb{F}}=6d-21b-10$ we see that $v_{\mathbb{F}}\geq -1$ which is equivalent to $b\leq\frac{2}{7}d-\frac{3}{7}$. Therefore the condition for getting $\mathcal{L}_{\mathbb{F}}$ non-special gives already that $v_{\mathbb{F}}\geq -1$.

From Theorem B we note again that $b>\frac{d}{4}$ confirms that $\hat{\mathcal{L}}_{\mathbb{F}}$ is non-special and $\hat{v}_{\mathbb{F}}\leq -1$.

Let us now consider $\hat{\mathcal{L}}_{\mathbb{P}}$: As above $\hat{\mathcal{L}}_{\mathbb{P}}$ is by induction non-special if $n-b$ is odd and $d-m_0\geq 11$. In the case $d-m_0\geq 11$ we force also $\hat{v}_{\mathbb{P}}\leq v(\mathcal{L})$, that is $b\leq\frac{6d-9}{21}$. In the case $d-m_0=10$ we conclude - exactly as above - that if $n-b$ is odd $\hat{\mathcal{L}}_{\mathbb{P}}$ is non-special or $\hat{\ell}_{\mathbb{P}}=-20(n-b)+5(d-6)-6$. Therefore we force $-20(n-b)+5(d-6)-6\leq -1$, that means $b\leq\frac{1}{4}(7-d)+n$. As we are in the case $v(\mathcal{L})\geq -1$ we have the equation $11d-21n-45\geq -1$ which means $n\leq\frac{11d-44}{21}$. It is enough to check the independence of all conditions on the base points in $\mathcal{L}$ for the highest possible number n of points. We fix this n and use a lower bound $\frac{11d-44}{21}-1$ of it. That means a sufficient condition for $\hat{\mathcal{L}}_{\mathbb{P}}$ to be non-special is $b\leq\frac{1}{4}(7-d)+\frac{11d-44}{21}-1=\frac{-113+23d}{84}$.

To fulfill all these conditions we need to have d large enough. All together this gives so far:

If $d-m_0\geq 11$ we are able to find a sufficient b if

$$\frac{2}{7}d-\frac{6}{7}-\frac{1}{4}d>2 \Longleftrightarrow d\geq 81.$$

If $d-m_0=10$ we are able to find a sufficient b if

$$\frac{23}{84}d-\frac{113}{84}-\frac{1}{4}d>2 \Longleftrightarrow d\geq 141.$$

In both cases we have that $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ are non-special and $v_{\mathbb{F}}\geq -1$. From $\hat{v}_{\mathbb{F}}\leq -1$ and from $v_{\mathbb{P}}=v-1-\hat{v}_{\mathbb{F}}$ we get immediately $v_{\mathbb{P}}\geq -1$. We have $v\geq\hat{v}_{\mathbb{P}}$. As $\hat{\mathcal{L}}_{\mathbb{F}}$ and $\hat{\mathcal{L}}_{\mathbb{P}}$ are non-special we are able to conclude the following two cases:

If $\hat{v}_{\mathbb{P}}\leq -1$ then

$$\hat{\ell}_{\mathbb{P}}+\hat{\ell}_{\mathbb{F}}=-2\leq v-1,$$

and if $\hat{v}_{\mathbb{P}}\geq -1$ then

$$\hat{\ell}_{\mathbb{P}}+\hat{\ell}_{\mathbb{F}}=\hat{v}_{\mathbb{P}}+\hat{\ell}_{\mathbb{F}}\leq v-1.$$

In both cases we are able to apply Lemma 6.2 and conclude that $\mathcal{L}$ is non-special.

$\square$

7. Proof of the Lemmas

Before starting the proofs we should take some time to explain the use of Quadratic Cremona Transformations for our purpose. We identify such a transformation with blowing up three general points and blowing down their connecting lines. Such a transformation is called to be *based on the three points*. Furthermore one can see by the blow-up and -down interpretation that a linear system $\mathcal{L}(d, m_0, m_1, m_2, m_3, \ldots, m_n)$ is transformed by a Cremona transformation based on the points p_0, p_1, p_2 to a system $\mathcal{L}(2d - m_0 - m_1 - m_2, d - m_1 - m_2, d - m_0 - m_2, d - m_0 - m_1, m_3, \ldots, m_n)$. If all involved numbers are non-negative (see [CM98]), the dimension and the virtual dimension of a system $\mathcal{L}$ do not change under Cremona transformations. In fact a (-1)-curve splitting off a system $\mathcal{L}$ is transformed again into a (-1)-curve, which splits off the transformed system. Therefore it is equivalent to examine a system $\mathcal{L}$ or its Cremona transformed for our purpose. We use suitable sequences of Cremona transformations in the following proofs to obtain systems which are already examined in previous papers.

Proof of the lemma of three base points 6.3:
This can be seen by direct computations with base points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$. Of course, the statement is also included in the result in [H89]. $\square$

Proof of the lemma of large multiplicities m_0 in p_0 6.4:
We consider the system $\mathcal{L}(d, m_0, 6^n)$. For the case of $m_0 \geq d - 7$ [CM98, Proposition 6.2., Corollary 6.3., Proposition 6.4.] give a classification of the special systems of this type. Comparing it with our list in Theorem B gives the statement. Now let $d \geq 25$. The strategy for the proof is to perform a sequence of Cremona transformations in order to get systems, which can be examined easier. Furthermore we apply the degeneration method again and use again Cremona transformations to prove regularity of some of the obtained systems.

case: $d - m_0 = 8$

Let $\mathcal{L} = \mathcal{L}(d, d-8, 6^n)$. We note that if we perform k Cremona transformations, based on p_0 and successively on two other base points of multiplicity 6, we obtain that it is now equivalent to consider the Cremona transformed system (for the strategy see [LU02]):

$$\mathcal{L} \sim \mathcal{L}(d - 4k, d - 8 - 4k, 6^{n-2k}, 2^{2k})$$

We set $d - 8 = 4t + \epsilon$ with $\epsilon \in \{0, 1, 2, 3\}$. And $n = 2q + \eta$ with $\eta \in \{0, 1\}$.

If $t \leq q$ we perform $k = t$ transformations on $\mathcal{L}(d, d-8, 6^n)$ based on p_0 and successively two other base points of multiplicity 6 and obtain

$$\mathcal{L} \sim \mathcal{L}(8 + \epsilon, \epsilon, 6^{n-2t}, 2^{2t}).$$

The system on the right hand side is of bounded multiplicity, that means all multiplicities are ≤ 6. Such systems are special if and only if they are (-1)-special by [Y03].

If $t > q$ we perform $k = q$ transformations on $\mathcal{L}(d, d-8, 6^n)$ again based on p_0 and successively two other base points of multiplicity 6 and obtain

$$\mathcal{L} \sim \mathcal{L}(d - 4q, d - 8 - 4q, 6^{\eta}, 2^{2q}).$$

If $\eta = 0$ we are in the case of quasi-homogeneous linear systems of multiplicity 2, here the main conjecture is true by [CM98].

If $\eta = 1$ we have to examine systems of the type $\mathcal{L} = \mathcal{L}(\delta, \delta - 8, 6, 2^{2q})$ with $\delta = d - 4q$. Now let us perform a $(2, b)$-degeneration and get the following systems:

$$\mathcal{L}_{\mathbb{P}} = \mathcal{L}(\delta - 2, \delta - 8, 6, 2^{2q-b}) \quad \mathcal{L}_{\mathbb{F}} = \mathcal{L}(\delta, \delta - 2, 2^b)$$

$$\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(\delta-3,\delta-8,6,2^{2q-b}) \quad \hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(\delta,\delta-1,2^{b})$$

If $v(\mathcal{L}) \leq -1$ we want to apply lemma 6.1.

By our classification Theorem B there is no (-1)-special system of the type $\mathcal{L}(d,d-8,6^n)$ if $d \geq 25$. That means we have to show that the system $\mathcal{L}$ is empty. To use 6.1 we have again to consider all the systems obtained by the degeneration as in the proof of the main theorem.

In a first step let us consider $\hat{\mathcal{L}}_{\mathbb{F}}$. As $\hat{\mathcal{L}}_{\mathbb{F}}$ is a quasi-homogeneous system of multiplicity $m=2$ we see in [CM98], that this system is never special. Then $\hat{v}_{\mathbb{F}} = 2\delta - 3b$ leads to a sufficient condition to get $\hat{\mathcal{L}}_{\mathbb{F}}$ empty. This condition is $b \geq \frac{2\delta+1}{3}$.

In a next step we want to find a sufficient condition to get $\mathcal{L}_{\mathbb{F}}$ non-special. This is true by [CM98] if b is odd. So let us force b to be odd as a sufficient condition for this case.

Now we consider $\mathcal{L}_{\mathbb{P}}$. We claim: $\mathcal{L}_{\mathbb{P}}$ is non-special.

To show the claim we apply at first a Cremona transformation based on the points of multiplicity $\delta-8$, 6 and on one point of multiplicity 2. This leads to the following system:

$$\mathcal{L}_{\mathbb{P}} \sim \mathcal{L}(\delta-4,\delta-10,4,2^{2q-b-1}).$$

Above we forced b to be odd, therefore we assume $2q-b-1 \geq 2$ (otherwise skip this step) is even. Now we apply successively $\frac{2q-b-1}{2}$ Cremona transformations, based in p_0 and two points of multiplicity 2. Therefore we see that we have the following equivalence:

$$\mathcal{L}_{\mathbb{P}} \sim \mathcal{L}(\delta-4+2q-b-1,\delta-10+2q-b-1,4^{2q-b}).$$

From $\delta = d-4q \geq 12+\epsilon$ we get by [S99, Theorem 2.1, Theorem 5.2] that this system is never special.

Finally we have to consider $\hat{\mathcal{L}}_{\mathbb{P}}$. Again we claim that $\hat{\mathcal{L}}_{\mathbb{P}}$ is never special.

We have by the above assumption that $2q-b$ is odd. At first we split off the line through the points of multiplicity $\delta-8$ and 6. As the virtual dimension doesn't change we get

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(\delta-4,\delta-9,5,2^{2q-b}).$$

Another Cremona transformation based in p_0, p_1 and one point of multiplicity 2 leads to the equivalence

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(\delta-6,\delta-11,3,2^{2q-b-1}).$$

Now as in the case of $\mathcal{L}_{\mathbb{P}}$ we apply another $\frac{2q-b-1}{2}$ Cremona transformations based in p_0 and successively in two points of multiplicity 2. We end up with the equivalence:

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(\delta-6+\frac{2q-b-1}{2},\delta-11+\frac{2q-b-1}{2},3^{2q-b}).$$

Now we are able to conclude with [CM98] - as we are in the case of a quasi-homogeneous system of multiplicity 3 - that this system is never special.

To apply 6.1 we have to find a sufficient condition for b to get $\hat{v}_{\mathbb{P}} \leq -1$, therefore it is sufficient to have $\hat{v}_{\mathbb{P}} - v(\mathcal{L}) \leq 0$, which is equivalent to $b \leq \delta$.

All together we find a sufficient b if $\delta - \frac{2\delta+1}{3} \geq 2 \iff \delta \geq 8$. As we have seen above we have already $\delta \geq 12+\epsilon$. This means we can apply Lemma 6.1 and conclude that $\mathcal{L}(d,d-8,6^n)$ is empty in the case $v(\mathcal{L}) \leq -1$.

Now we have to consider the case $v(\mathcal{L}) \geq -1$. Here we want to apply the Lemma 6.2.

As in the case $v(\mathcal{L}) \leq -1$ we can always find a b such that all the systems obtained by the above $(2,b)$-degeneration are non-special. Let us choose such a b like above and then consider the systems $\mathcal{L}_{\mathbb{P}}$, $\hat{\mathcal{L}}_{\mathbb{P}}$, $\mathcal{L}_{\mathbb{F}}$ and $\hat{\mathcal{L}}_{\mathbb{F}}$. From $v_{\mathbb{P}} = v(\mathcal{L}) - \hat{v}_{\mathbb{F}} - 1$, $\hat{v}_{\mathbb{F}} \leq -1$ and $v(\mathcal{L}) \geq -1$ we conclude $v_{\mathbb{P}} \geq v(\mathcal{L}) \geq -1$. A direct computation gives $v_{\mathbb{F}} \geq -1$.

As the inequality $\hat{v}_{\mathbb{P}} \leq v(\mathcal{L})$ is also fulfilled we get $\hat{\ell}_{\mathbb{P}} \leq v(\mathcal{L})$. Therefore we can apply Lemma 6.2 and conclude that $\mathcal{L}(d, d-8, 6^n)$ is non-special.

case: $d - m_0 = 9$

Let $\mathcal{L} = \mathcal{L}(d, d-9, 6^n)$. We note as above that if we perform k Cremona transformations, based on p_0 and successively on two other base points of multiplicity 6, we obtain that:

$$\mathcal{L} \sim \mathcal{L}(d-3k, d-9-3k, 6^{n-2k}, 3^{2k})$$

We set $d - 9 = 3t + \epsilon$ with $\epsilon \in \{0, 1, 2\}$. And $n = 2q + \eta$ with $\eta \in \{0, 1\}$.

If $t \leq q$ we perform $k = t$ transformations on $\mathcal{L}(d, d-9, 6^n)$ based on m_0 and successively on two other base points of multiplicity 6 and obtain

$$\mathcal{L} \sim \mathcal{L}(9 + \epsilon, \epsilon, 6^{n-2t}, 3^{2t}).$$

Then the system on the right hand side is of bounded multiplicity, that means all multiplicities are ≤ 6. As mentioned above such systems are special if and only if they are (-1)-special by [Y03].

If $t > q$ we perform $k = q$ transformations on $\mathcal{L}(d, d-9, 6^n)$ and obtain

$$\mathcal{L} \sim \mathcal{L}(d-3q, d-9-3q, 6^{\eta}, 3^{2q}).$$

If $\eta = 0$ we are in the case of quasi-homogeneous linear systems of multiplicity 3, here the main conjecture is true by [CM98].

If $\eta = 1$ we have to examine systems of the type $\mathcal{L}(\delta, \delta-9, 6, 3^{2q})$ with $\delta = d-3q$. If $\delta < 15$ we are in the case of systems of bounded multiplicity where the main conjecture holds by [Y03]. So we can assume $\delta \geq 15$. Also we can assume $q \geq 1$ (otherwise the statement is clear). Now let us perform a $(3, b)$-degeneration and get the following systems:

$$\begin{array}{ll} \mathcal{L}_{\mathbb{P}} = \mathcal{L}(\delta-3, \delta-9, 6, 3^{2q-b}) & \mathcal{L}_{\mathbb{F}} = \mathcal{L}(\delta, \delta-3, 3^b) \\ \hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(\delta-4, \delta-9, 6, 3^{2q-b}) & \hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(\delta, \delta-2, 3^b) \end{array}$$

If $v(\mathcal{L}) \leq -1$ we again want to apply Lemma 6.1.

So let as go through all the systems from the above $(3, b)$-degeneration and search for sufficient conditions on b to apply Lemma 6.1.

Let us consider $\hat{\mathcal{L}}_{\mathbb{F}}$ at first. Here it is sufficient to choose $b > \frac{\delta}{2}$ to get this system non-special by [CM98] and $\hat{\ell}_{\mathbb{F}} = -1$.

In a next step consider $\mathcal{L}_{\mathbb{F}}$. By [CM98] this is non-special if b is odd.

Then we force (to apply 6.1) $\hat{v}_{\mathbb{P}} \leq v$. This is fulfilled if $b \leq \frac{2\delta-1}{3}$.

Now let us consider $\mathcal{L}_{\mathbb{P}}$. We claim that this system is never special. To see that let us perform Cremona transformation based on the points of multiplicity $\delta - 9$, 6 and 3. We obtain:

$$\mathcal{L}_{\mathbb{P}} \sim \mathcal{L}(\delta-6, \delta-12, 3^{2q-b-1})$$

These systems are always regular by [CM98] as we have δ high enough.

A little bit more complicated is the case of $\hat{\mathcal{L}}_{\mathbb{P}}$. We are searching for a sufficient condition on b to get $\hat{\mathcal{L}}_{\mathbb{P}}$ empty. We want to show, that $\hat{\mathcal{L}}_{\mathbb{P}}$ is never special. Then we get the condition simply be choosing b such that $\hat{v}_{\mathbb{P}} \leq -1$ (fulfilled by $\hat{v}_{\mathbb{P}} \leq v$).

First of all we split off a line through p_0, the point of multiplicity $m_0 = \delta - 9$, and the point of multiplicity 6. Therefore we obtain

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(\delta-5, \delta-10, 5, 3^{2q-b}).$$

as the virtual dimension doesn't change in that case. If $2q - b > 0$ applying a further Cremona transformation based in the points of multiplicity $\delta - 10$, 5 and one point of multiplicity 3 gives

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(\delta - 8, \delta - 13, 2, 3^{2q-b-1}).$$

Note that $2q - b - 1$ is an even number, as b is odd. We apply now successively Cremona transformations, based on the point p_0 and on two other points of multiplicity 3. It is better again to consider two different cases.

At first assume $\delta - 13 \geq \frac{2q-b-1}{2}$. Then we get

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(\delta - 8 - \frac{2q-b-1}{2}, \delta - 13 - \frac{2q-b-1}{2}, 2^{2q-b}).$$

By [CM98] such a system is never special.

Secondly assume $\delta - 13 < \frac{2q-b-1}{2}$. Let $m = \delta - 13$. Then after m such transformations we obtain:

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(5, 2^{2m+1}, 3^{2q-b-1-2m}).$$

Again splitting off a line through two points of multiplicity 3 (virtual dimension does not change) gives:

$$\hat{\mathcal{L}}_{\mathbb{P}} \sim \mathcal{L}(4, 2^{2(m+1)+1}, 3^{2q-b-1-2(m+1)}).$$

Now two if $2q - b - 1 - 2(m + 1) \geq 2$ splitting off lines gives that $\mathcal{L}_{\mathbb{P}}$ is empty. Secondly if $2q - b - 1 - 2(m + 1) = 0$ we have also by [CM98] that the system is empty (as $m \geq 2$ by assumption that $\delta \geq 15$).

Taking into account all our conditions on b we require

$$\frac{2\delta - 1}{3} - \frac{\delta}{2} > 2 \Longleftrightarrow \delta \geq 15.$$

Finally applying Lemma 6.1 gives that the system $\mathcal{L} = \mathcal{L}(\delta, \delta - 9, 6, 3^{2q})$ is empty in the case $v \leq -1$.

Now we have to consider the case $v(\mathcal{L}) \geq -1$. We want to apply Lemma 6.2.

As in the case of $v \leq -1$ we get that all the systems $\mathcal{L}_{\mathbb{F}}$, $\hat{\mathcal{L}}_{\mathbb{F}}$, $\mathcal{L}_{\mathbb{P}}$ and $\hat{\mathcal{L}}_{\mathbb{P}}$ are non-special and $\hat{\ell}_{\mathbb{F}} = -1$ and $\hat{v}_{\mathbb{P}} \leq v$ with a suitable b for the degeneration.

That means here $\hat{\ell}_{\mathbb{P}} \leq v$ and we can apply Lemma 6.2 and conclude that $v(\mathcal{L}) = \ell$, that means $\mathcal{L} = \mathcal{L}(\delta, \delta - 9, 6, 3^{2q})$ is regular.

This finally completes our proof for the case of multiplicities $m_0 = d - 8$ and $m_0 = d - 9$.

□

Proof of the lemma of low degrees 6.5:
The main tool for this proof is a computer program which uses $(5, b)$- and $(6, b)$-degenerations of the plane in order to prove that certain non-(-1)-special systems are non-special. This algorithm is given by Laface and Ugaglia in [LU02]. We implemented this algorithm in *Singular* (see [Sing]). Furthermore to treat the cases where the degeneration-method fails we implemented a method used by Yang in [Y03]. This method specializes the base points on a line and moves them to infinity. Then it is easier to check if the given conditions on the base points are independent. If this is still the case it proves regularity of a given system.
Below we list only the cases in which the program fails. All these but 10 cases are solved by ad-hoc methods (mainly Cremona transformations). The remaining 10 cases we computed directly with *Singular* in characteristic 32003. One can see that this implies then regularity in characteristic 0, too.

$d - m_0$	system	dimension	method
8	$\mathcal{L} = \mathcal{L}(8, 0, 6^3)$	-1	3-point lemma
8	$\mathcal{L} = \mathcal{L}(9, 1, 6^3)$	-1	splitting off lines
14	$\mathcal{L} = \mathcal{L}(14, 0, 6^6)$	-1	Cremona and splitting off lines
13	$\mathcal{L} = \mathcal{L}(14, 1, 6^6)$	-1	as $\mathcal{L}(14, 0, 6^6)$ is empty
12	$\mathcal{L} = \mathcal{L}(14, 2, 6^6)$	-1	as $\mathcal{L}(14, 0, 6^6)$ is empty
11	$\mathcal{L} = \mathcal{L}(14, 3, 6^6)$	-1	as $\mathcal{L}(14, 0, 6^6)$ is empty
10	$\mathcal{L} = \mathcal{L}(14, 4, 6^6)$	-1	as $\mathcal{L}(14, 0, 6^6)$ is empty
8	$\mathcal{L} = \mathcal{L}(14, 6, 6^5)$	-1	as $\mathcal{L}(14, 0, 6^6)$ is empty
15	$\mathcal{L} = \mathcal{L}(15, 0, 6^7)$	-1	Cremona
15	$\mathcal{L} = \mathcal{L}(15, 0, 6^6)$	> -1	as $\mathcal{L}(15, 3, 6^6)$ is regular
14	$\mathcal{L} = \mathcal{L}(15, 1, 6^6)$	> -1	as $\mathcal{L}(15, 3, 6^6)$ is regular
13	$\mathcal{L} = \mathcal{L}(15, 2, 6^6)$	> -1	as $\mathcal{L}(15, 3, 6^6)$ is regular
12	$\mathcal{L} = \mathcal{L}(15, 3, 6^6)$	> -1	Cremona and [CM98]
11	$\mathcal{L} = \mathcal{L}(15, 4, 6^6)$	-1	Cremona and splitting off lines
10	$\mathcal{L} = \mathcal{L}(15, 5, 6^6)$	-1	as $\mathcal{L}(15, 4, 6^6)$ is empty
9	$\mathcal{L} = \mathcal{L}(15, 6, 6^6)$	-1	as $\mathcal{L}(15, 4, 6^6)$ is empty
9	$\mathcal{L} = \mathcal{L}(15, 6, 6^5)$	> -1	as $\mathcal{L}(15, 0, 6^6)$ is regular
8	$\mathcal{L} = \mathcal{L}(15, 7, 6^5)$	> -1	Cremona and [CM98]
16	$\mathcal{L} = \mathcal{L}(16, 0, 6^8)$	-1	as $\mathcal{L}(16, 3, 6^7)$ is empty
16	$\mathcal{L} = \mathcal{L}(16, 0, 6^7)$	> -1	as $\mathcal{L}(16, 2, 6^7)$ is regular
15	$\mathcal{L} = \mathcal{L}(16, 1, 6^7)$	> -1	as $\mathcal{L}(16, 2, 6^7)$ is regular
14	$\mathcal{L} = \mathcal{L}(16, 2, 6^7)$	> -1	Cremona and [CM98]
13	$\mathcal{L} = \mathcal{L}(16, 3, 6^7)$	-1	Cremona and splitting off lines
12	$\mathcal{L} = \mathcal{L}(16, 4, 6^7)$	-1	as $\mathcal{L}(16, 3, 6^7)$ is empty
11	$\mathcal{L} = \mathcal{L}(16, 5, 6^7)$	-1	as $\mathcal{L}(16, 3, 6^7)$ is empty
10	$\mathcal{L} = \mathcal{L}(16, 6, 6^7)$	-1	as $\mathcal{L}(16, 3, 6^7)$ is empty
10	$\mathcal{L} = \mathcal{L}(16, 6, 6^6)$	> -1	as $\mathcal{L}(16, 2, 6^7)$ is regular
9	$\mathcal{L} = \mathcal{L}(16, 7, 6^6)$	-1	Cremona and splitting off lines
8	$\mathcal{L} = \mathcal{L}(16, 8, 6^6)$	-1	as $\mathcal{L}(16, 7, 6^6)$ is empty
17	$\mathcal{L} = \mathcal{L}(17, 0, 6^8)$	> -1	as $\mathcal{L}(17, 1, 6^8)$ is regular
16	$\mathcal{L} = \mathcal{L}(17, 1, 6^8)$	> -1	Cremona
15	$\mathcal{L} = \mathcal{L}(17, 2, 6^8)$	-1	Cremona and splitting off lines
11	$\mathcal{L} = \mathcal{L}(17, 6, 6^7)$	> -1	as $\mathcal{L}(17, 1, 6^8)$ is regular
10	$\mathcal{L} = \mathcal{L}(17, 7, 6^7)$	-1	Cremona and splitting off lines
9	$\mathcal{L} = \mathcal{L}(17, 8, 6^7)$	-1	as $\mathcal{L}(17, 7, 6^7)$ is empty
8	$\mathcal{L} = \mathcal{L}(18, 10, 6^7)$	-1	Cremona and splitting off lines
19	$\mathcal{L} = \mathcal{L}(19, 0, 6^{10})$	-1	[CM00]
18	$\mathcal{L} = \mathcal{L}(19, 1, 6^{10})$	-1	as $\mathcal{L}(19, 0, 6^{10})$ is empty
17	$\mathcal{L} = \mathcal{L}(19, 2, 6^{10})$	-1	as $\mathcal{L}(19, 0, 6^{10})$ is empty
15	$\mathcal{L} = \mathcal{L}(19, 4, 6^9)$	> -1	as $\mathcal{L}(19, 5, 6^9)$ is regular
14	$\mathcal{L} = \mathcal{L}(19, 5, 6^9)$	> -1	regular by [Y03]
13	$\mathcal{L} = \mathcal{L}(19, 6, 6^9)$	-1	as $\mathcal{L}(19, 0, 6^{10})$ is empty
12	$\mathcal{L} = \mathcal{L}(19, 7, 6^9)$	-1	as $\mathcal{L}(19, 0, 6^{10})$ is empty
9	$\mathcal{L} = \mathcal{L}(19, 10, 6^7)$	> -1	Cremona and [CM00]
8	$\mathcal{L} = \mathcal{L}(19, 11, 6^7)$	-1	Cremona and splitting off lines
12	$\mathcal{L} = \mathcal{L}(20, 8, 6^9)$	> -1	direct computation with [Sing] in char= 32003

$d-m_0$	system	dimension	method
11	$\mathcal{L}=\mathcal{L}(20,9,6^9)$	-1	Cremona and [LU02]
8	$\mathcal{L}=\mathcal{L}(20,12,6^7)$	>-1	Cremona and [CM00]
11	$\mathcal{L}=\mathcal{L}(21,10,6^9)$	>-1	Cremona and [Y03]
10	$\mathcal{L}=\mathcal{L}(21,11,6^9)$	-1	Cremona and [Y03]
9	$\mathcal{L}=\mathcal{L}(21,12,6^8)$	>-1	Cremona and [CM00]
8	$\mathcal{L}=\mathcal{L}(21,13,6^8)$	-1	Cremona, splitting off lines and [CM00]
22	$\mathcal{L}=\mathcal{L}(22,0,6^{13})$	>-1	as $\mathcal{L}(22,1,6^{13})$ is regular
21	$\mathcal{L}=\mathcal{L}(22,1,6^{13})$	>-1	[Y03]
20	$\mathcal{L}=\mathcal{L}(22,2,6^{13})$	-1	[Y03]
19	$\mathcal{L}=\mathcal{L}(22,3,6^{13})$	-1	as $\mathcal{L}(22,2,6^{13})$ is empty
16	$\mathcal{L}=\mathcal{L}(22,6,6^{12})$	>-1	as $\mathcal{L}(22,1,6^{13})$ is regular
15	$\mathcal{L}=\mathcal{L}(22,7,6^{12})$	-1	direct computation with [Sing] in char $=32003$
13	$\mathcal{L}=\mathcal{L}(22,9,6^{11})$	-1	$''$
11	$\mathcal{L}=\mathcal{L}(22,11,6^{10})$	-1	Cremona and [Y03]
10	$\mathcal{L}=\mathcal{L}(22,12,6^{10})$	-1	as $\mathcal{L}(22,11,6^{10})$ is empty
10	$\mathcal{L}=\mathcal{L}(22,12,6^9)$	>-1	Cremona and [S99]
9	$\mathcal{L}=\mathcal{L}(22,13,6^9)$	-1	Cremona and splitting off lines
8	$\mathcal{L}=\mathcal{L}(22,14,6^9)$	-1	as $\mathcal{L}(22,13,6^9)$ is empty
12	$\mathcal{L}=\mathcal{L}(23,11,6^{11})$	>-1	direct computation with [Sing] in char $=32003$
10	$\mathcal{L}=\mathcal{L}(23,13,6^{10})$	-1	Cremona and [S99]
9	$\mathcal{L}=\mathcal{L}(23,14,6^9)$	>-1	Cremona and [CM00]
8	$\mathcal{L}=\mathcal{L}(23,15,6^9)$	-1	Cremona and splitting off lines
10	$\mathcal{L}=\mathcal{L}(24,14,6^{10})$	>-1	Cremona and [CM00]
9	$\mathcal{L}=\mathcal{L}(24,15,6^{10})$	-1	Cremona and [Y03]
8	$\mathcal{L}=\mathcal{L}(24,16,6^{10})$	-1	as $\mathcal{L}(24,15,6^{10})$ is empty
13	$\mathcal{L}=\mathcal{L}(25,12,6^{13})$	-1	direct computation with [Sing] in char $=32003$
10	$\mathcal{L}=\mathcal{L}(25,15,6^{11})$	-1	Cremona and [Y03]
12	$\mathcal{L}=\mathcal{L}(26,14,6^{13})$	-1	direct computation with [Sing] in char $=32003$
10	$\mathcal{L}=\mathcal{L}(29,19,6^{13})$	>-1	direct computation with [Sing] in char $=32003$
13	$\mathcal{L}=\mathcal{L}(31,18,6^{17})$	-1	$''$
10	$\mathcal{L}=\mathcal{L}(31,21,6^{14})$	>-1	Cremona and [S99]
10	$\mathcal{L}=\mathcal{L}(38,28,6^{18})$	-1	Cremona and [S99]
13	$\mathcal{L}=\mathcal{L}(40,27,6^{23})$	-1	direct computation with [Sing] in char $=32003$
10	$\mathcal{L}=\mathcal{L}(40,30,6^{19})$	-1	$''$
10	$\mathcal{L}=\mathcal{L}(46,36,6^{22})$	-1	Cremona and [S99]

$\square$

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